Chebyshev expansion for accurate and efficient 2D PSP bistatic Point Target Spectrum

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***Abstract*—The effectiveness of frequency domain processing algorithms in bistatic SAR depends on the accuracy of approximations used for the derivation of the bistatic point target spectrum. This paper presents a modified bistatic point target spectrum based on the 2 dimensional principle of stationary phase with the accuracy improved by the use of a Chebyshev polynomial approximation. The performance of the new method is compared to the conventional Taylor series approximation approach in the approximation of the range-azimuth frequencies coupling variable. The new approach is shown to provide a more accurate approximation reducing the phase error. The accuracy improvement is shown to yield a more accurate spectrum that can be exploited in bistatic SAR focusing algorithms, improving their accuracy and efficiency.**

***Index Terms*—Bistatic SAR, point target spectrum, Chebyshev approximation**

I. Introduction

Bistatic SAR operates with a separate transmitter and receiver introducing new characteristics compared to traditional monostatic SAR systems [1]. Some of the advantages of the bistatic configuration include i) the reduction of vulnerability of the system in military applications with the ability of having the transmitter located at save distances from a hostile area ii) the capability for the bistatic system to be employed for imaging in the flight direction or backwards in flight assistance systems iii) reduction of costs iv) measurement of the bistatic clutter characteristic and v) the reduction of the dihedral and polyhedral effects in urban areas improving the images quality. However the bistatic flight configuration poses two critical technological challenges. The first involves syncronization of the transmitter and the receiver both in space and time. A good solution for this issue was developed in [2]. The second critical challenge of the bistatic SAR configuration is the requirement for an approximation of the bistatic point target spectrum which is significantly different from the monostatic case. This is principally due to the bistatic slant range function that is characterized by the sum of two hyperbolas [1] rather than a single hyperbola in the monostatic case. This double squared root function makes the derivation of a signal spectrum model for focussing algorithms in the frequency domain much more challenging than in the monostatic case. Different approaches have been proposed to address this problem, some numerical like NuSAR [3] and the Dip Move Out technique [4], some mathematical such as the Loffeld's bistatic formula [5] the 2D Principle of Stationary Phase (PSP) [6] and the Method of Series Reversion [7], [8]. The Point Target Spectrum (PTS) proposed in [6] is probably the most accurate and robust PTS able to work in the azimuth variant case. In our previous work [9] the accurancy of the bistatic point target spectrum based on the method of Series reversion [7] was improved using the Chebyshev polynomial series expansion. In this work we propose a similar approach in order to improve the 2D PSP based PTS [6] replacing the Taylor expansion. We will present the improvements using the Chebyshev polynomial approximation [10] to be used for the decoupling of the range-azimuth frequency components. The proposed approach replaces the coefficients of the polynomial Taylor series expansion of the azimuth-range frequency coupling term with the ones obtained using the Chebyshev polynomial approximation. The obtained coefficients are then used to obtain an analytical formulation of the bistatic point target spectrum as in [6].

The remainder of paper is organized as follows. In section II the bistatic SAR signal model and the 2D bistatic point target spectrum are discussed. In section III the new method of using Chebyshev polynomials is developed deriving the new bistatic Point Target Spectrum. Section IV presents comparative results that indicate the superior performance obtained to those obtained using the original method [6].

II. Bistatic Signal Model and 2D PSP Point Target Spectrum

In a bistatic SAR the transmitter and the receiver can have different velocities, altitudes and flight paths, leading to the possibility of having different acquisition configurations. The simplest case is when both transmitter and receiver have the same velocity and parallel flight paths, while a more complicated configuration exists when the platforms have different velocities and non parallel flight paths. As in the monostatic case [11] the area to be imaged is a collection of point scatterers. This implies that it is sufficient to analyse the scene using the response of an arbitrary point scatterer and then consider the superposition of the echoes to obtain the focused image. The received bistatic signal model can be written as [6]:

$$s(\tau, t, \tau_{0R}, R_{0R}) = \sigma(\tau_{0R}, R_{0R}) p\left(t - \frac{R_R(\tau) + R_T(\tau)}{c}\right) \times exp\left[-j2\pi \frac{R_R(\tau) + R_T(\tau)}{\lambda}\right] w(\tau - \tau_{cb}) \tag{1}$$

where τ is the slow time, $\sigma(\tau_{0R}, R_{0R})$ is the complex backscatterer coefficient of the point target located at (τ_{0R}, R_{0R}), λ is the carrier wavelength , $p(.)$ is the range envelope and $w(.)$ is the composite antenna pattern of the transmitter and receiver centred in the azimuth time τ_{cb}. $R_R(\tau)$ and $R_T(\tau)$ are the instantaneous receiver and transmitter slant ranges to the point target.

$$\begin{aligned} R_R(\tau) &= \sqrt{R_{0R}^2 + (\tau - \tau_{0R})^2 v_R^2} \\ R_T(\tau) &= \sqrt{R_{0T}^2 + (\tau - \tau_{0T})^2 v_T^2} \end{aligned} \tag{2}$$

where R_{0R} and R_{0T} are the closest slant ranges from the receiver and transmitter to the point target, v_R and v_T are the receiver and transmitter velocity respectively. From (2) and (1) it can be seen that the signal model is modulated in the slow time by a double squared root (DSR) function given by the sum of the istantaneous slant ranges. This makes the derivation of an accurate bistatic point target spectrum a difficult problem to solve.

In [6] a bistatic SAR point target spectrum using the two-dimensional principle of stationary phase was proposed. It was demonstrated the accuracy of this BPTS with simulated and real data. The expression of the point target spectrum obtained in [6] is:

$$\begin{aligned} \Psi_B(f_\tau, f, R_{0R}) = \pi\beta_a\tau_{0R}f_\tau + 2\pi(p_{01} + p_{11}R_{0R})k_T f_\tau + \\ 2\pi\left[\frac{R_{0R}}{c}F_R + \frac{R_{0T}}{c}F_T\right] + \bar{\Phi}_{RCM}(f) + \Phi_{Res}(\tau_{0R}, R_{0R}) \end{aligned} \tag{3}$$

where $P(\tau_{0R}, R_{0R})$, τ_{0R} is the zero-Doppler time of the receiver, the coefficients p_{ij} are obtained using the geometrical image transformation of τ_{0T} to obtain as a function of τ_{0R} and R_{0R}. β_a is the scaling factor in the azimuth domain defined as $\beta_a = k_R + p_{12}k_T$, with k_R and k_T denoting the receiver and transmitter azimuth modulation rate. $\bar{\Phi}_{RCM}(f)$ and $\Phi_{Res}(\tau_{0R}, R_{0R})$ are a component of the RCM and the residual phase, these two are not of our direct interest and their expression can be found in [6]. To represent the PTS in a series of power of the range frequency and separate the different components of the range-azimuth coupling a Taylor series approximation of the hyperbolic frequencies coupling variable is introduced.

$$\begin{aligned} F_R(f) &= \sqrt{(f + f_0)^2 - \left(\frac{cf_{\tau_R}}{v_R}\right)^2} \\ F_T(f) &= \sqrt{(f + f_0)^2 - \left(\frac{cf_{\tau_T}}{v_T}\right)^2} \end{aligned} \tag{4}$$

where f is the range frequency, f_0 is the carrier frequency, c is the speed of light, v_r and v_t are the platform velocities and f_{τ_R} and f_{τ_T} are the azimuth frequency variables representing the contribute of the range equations of the receiver and transmitter to the instantaneous Doppler frequency f_τ.

As we stated before the expression of F_R and F_T must be expanded in a power series to separate the different contributions. In [6] the Taylor series is used leading to:

$$\begin{aligned} F_R(f) &\approx D_R f_0 + \frac{(1 - \mu_{R1}\mu_{R2})}{D_R} f - \frac{(\mu_{R1} - \mu_{R2})^2}{2f_0 D_R^3} f^2 \\ F_T(f) &\approx D_T f_0 + \frac{(1 - \mu_{T1}\mu_{T2})}{D_T} f - \frac{(\mu_{T1} - \mu_{T2})^2}{2f_0 D_T^3} f^2 \end{aligned} \tag{5}$$

where :

$$\begin{aligned} & D_R = \sqrt{1 - \mu_{R1}^2} \qquad D_T = \sqrt{1 - \mu_{T1}^2} \\ & \mu_c = \frac{k_T v_R sin\theta_{SR} - k_R v_T sin\theta_{ST}}{\lambda} \\ & \times (p_{01} + p_{11}R_{0R} + p_{12}\tau_{0R}) \\ & + 2\pi\frac{k_T v_R sin\theta_{SR} - k_R v_T sin\theta ST}{\lambda} \\ & \mu_{R1} = \frac{\lambda}{v_R}(k_R f_\tau + \mu_c) \qquad \mu_{R2} = \frac{\lambda}{v_R}\mu_c \\ & \mu_{T1} = \frac{\lambda}{v_T}(k_T f_\tau - \mu_c) \qquad \mu_{T2} = -\frac{\lambda}{v_T}\mu_c \end{aligned} \tag{6}$$

θ_{SR} and θ_{ST} are the receiver and transmitter squint angle at the composite beam center. Then substituting (5) in (3) it is possible to obtain a form of the PTS useful to develop focussing algorithms.

$$\begin{aligned} \Psi_B(f_\tau, f, R_{0R}) = \Phi_{RC}(f_\tau, f) + \Phi_{RCM}(f_\tau, f, R_{0R}) + \\ \Phi_{AC}(f_\tau, R_{0R}) + \Phi_{AS}(f_\tau) \end{aligned} \tag{7}$$

with:

$$\Phi_{RC}(f_\tau, f) \approx \pi\frac{f^2}{K_r} - \pi\frac{f^2}{K_{SRC}} \tag{8}$$

$$\Phi_{RCM}(f_\tau, f, R_{0R}) = \frac{2\pi}{c}\left[\frac{R_{0R}}{D_R} + \frac{R_{0T}}{D_T}\right] f \tag{9}$$

$$\begin{aligned} \Phi_{AC}(f_\tau, R_{0R}) = 2\pi(p_{01} + p_{11}R_{0R})k_T f_\tau \\ + \frac{2\pi}{\lambda}(R_{0R}D_R + R_{0T}D_T) \end{aligned} \tag{10}$$

$$\Phi_{AS}(f_\tau) = 2\pi\beta_a\tau_{0R}f_\tau \tag{11}$$

$$\begin{aligned} \frac{1}{K_{SRC}} = \Bigg[R_{RR}\frac{(\mu_{R1} - \mu_{R2})^2}{cf_0 D_R^3} + \\ R_{RT}\frac{(\mu_{T1} - \mu_{T2})^2}{cf_0 D_T^3} \Bigg] \end{aligned} \tag{12}$$

R_{RR} and R_{RT} are the reference slant range of the transmitter and the receiver respectively.

The spectrum in (7) is the main result in [6] that resulted to be accurate and was tested with real data, however it can still improved increasing its approximation accuracy. In the next section our approach replacing the Taylor approximation with the Chebyshev one is proposed.

III. POLYNOMIAL APPROXIMATIONS USING CHEBYSHEV POLYNOMIALS

Weierstrass theorem [12] suggests that polynomial approximation of a continuous function in an interval $[a, b]$ can be always performed. A common approach in obtaining an approximating polynomial is to minimize the square residual for the approximation of a function. This minimization finds a solution for the least squares problem, which is solved using an interpolating polynomial family of orthogonal polynomials. The use of this kind of polynomial approximation facilitates convergence to the target function as the approximation order increases. The orthogonal Chebyshev polynomials allow us to obtain an approximation which minimizes the error in the sense of the least squares [12] and the infinity norm. Another property is to reduce the so called Runge phenomenon, meaning that the maximum approximation error is bounded. In general the Chebyshev polynomials approximation offers closer approximation to the minimax polynomial which represent the polynomial with the smallest maximum deviation from the true function [10]. In [13] it was shown that the use of Chebyshev polynomials to approximate functions outperforms the Taylor series expansion. For these reasons Chebyshev approximation was chosen in this work to replace the Taylor series used in [6]. The properties of the Chebyshev polynomial approximation lead to a better approximation of the phase history of the signal allowing the design of a more accurate matched filter to improve the image quality, this result is achieved by the reduction of the overall error. In our previous work [9] the accuracy of the bistatic point target spectrum based on the method of Series reversion [7] was improved using the Chebyshev polynomial series expansion. In this work we propose a similar approach in order to improve the 2D PSP based PTS replacing the Taylor expansion in (5). We then approximate (4) using the Chebyshev polynomial of the first kind stopping the approximation at the second order as in (5):

$$\begin{aligned}
F_R(f) \approx & \sum_{j=1}^{2} g_{jR} T_j(f) - \frac{1}{2} g_{0R} = -\frac{1}{2} g_{0R} + \\
& g_{1R} T_1(f) + g_{2R} T_2(f) = \\
& -\left(\frac{1}{2} g_{0R} + g_{2R}\right) + g_{1R} f + 2 g_{2R} f^2 = \\
& \hat{F}_{0R} + g_{1R} f + 2 g_{2R} f^2 \\
F_T(f) \approx & \sum_{j=1}^{2} g_{jT} T_j(f) - \frac{1}{2} g_{0T} = -\frac{1}{2} g_{0T} + \\
& g_{1T} T_1(f) + g_{2T} T_2(f) = \\
& \left(-\frac{1}{2} g_{0T} + g_{2T}\right) + g_{1T} f + 2 g_{2T} f^2 \\
& \hat{F}_{0T} + g_{1T} f + 2 g_{2T} f^2
\end{aligned} \tag{13}$$

the Chebyshev coefficients g_{jR} and g_{jT} are computed as:

$$\begin{aligned}
g_{0R} &= \frac{1}{3} \sum_{k=0}^{2} F_R(f_k) T_0(f_k) \quad g_{jR} = \frac{2}{3} \sum_{k=0}^{2} F_R(f_k) T_j(f_k) \\
g_{0T} &= \frac{1}{3} \sum_{k=0}^{2} F_T(f_k) T_0(f_k) \quad g_{jT} = \frac{2}{3} \sum_{k=0}^{2} F_T(f_k) T_j(f_k)
\end{aligned} \tag{14}$$

where f_k are the Chebyshev nodes obtained as:

$$f_k = \frac{f_{min} + f_{max}}{2} - \left[\frac{f_{max} + f_{min}}{2} cos\left(\left(\frac{2k+1}{6}\pi\right)\right)\right] \tag{15}$$

with f_{min} and f_{max} representing the maximum and the minimum of the range Doppler spectrum. The term in (3) that will be affected from our approximation is the third term $\chi(f, f_\tau) = \pi \left[\frac{R_{0R}}{c} F_R + \frac{R_{0T}}{c} F_T\right]$, that becomes:

$$\begin{aligned}
\chi(f, f_\tau) \approx & 2\pi \left[\frac{R_{0R}}{c} \hat{F}_R + \frac{R_{0T}}{c} \hat{F}_T\right] = \\
& 2\pi \left[\frac{R_{0R}}{c} (\hat{F}_0 R + g_{1R} f + 2 g_{2R} f^2)\right. \\
& \left. + \frac{R_{0T}}{c} (\hat{F}_{0T} + g_{1T} f + 2 g_{2T} f^2)\right] = \\
& \frac{2\pi}{c} (R_{0R} \hat{F}_{0R} + R_{0T} \hat{F}_{0T}) + \\
& \frac{2\pi f}{c} (R_{0R} g_{1R} + R_{0T} g_{1T}) + \\
& \frac{4\pi f^2}{c} (R_{0R} g_{2R} + R_{0T} g_{2T})
\end{aligned} \tag{16}$$

From (17) the different contributes can be separated and grouped with the remaining from (3) as:

$$\begin{aligned}
\hat{\Phi}_{RC}(f_\tau, f) &= \frac{\pi f^2}{k_R} + \frac{4\pi f^2}{c} (R_{0R} g_{2R} + R_{0T} g_{2T}) \\
\hat{\Phi}_{AC}(f_\tau, R_{0R}) &= 2\pi (p_{01} + p_{11} R_{0T}) k_T f_\tau \\
& \quad + \frac{2\pi}{c} (R_{0R} \hat{F}_{0R} + R_{0T} \hat{F}_{0T}) \\
\hat{\Phi}_{RCM}(f_\tau, f) &= \frac{2\pi f}{c} (R_{0R} g_{1R} + R_{0T} g_{1T}) + \bar{\Phi}_{RCM}(f)
\end{aligned} \tag{17}$$

The resulting bistatic point target spectrum is then :

$$\begin{aligned}
\hat{\Psi}_B(f_\tau, f, R_{0R}) = & \hat{\Phi}_{RC}(f_\tau, f) + \hat{\Phi}_{RCM}(f_\tau, f, R_{0R}) + \\
& \hat{\Phi}_{AC}(f_\tau, R_{0R}) + \Phi_{AS}(f_\tau)
\end{aligned} \tag{18}$$

In the next section the results on the phase error due to the approximation using Chebyshev and Taylor polynomials for airborne and space-borne configurations are shown.

IV. RESULTS

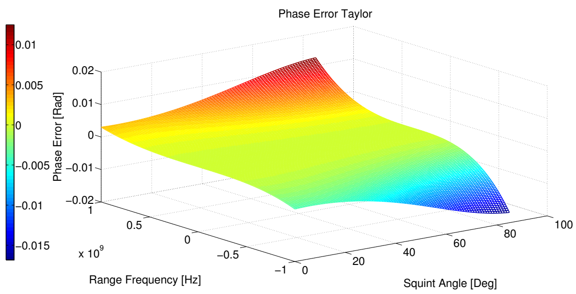
In this section we present the results obtained simulating both airborne and space-borne bistatic SAR configurations. The results refers to the resulting approximation phase error for Taylor and Chebyshev approaches. In Table I the two

Table I: Simulation parameters for the airborne and spaceborne configurations

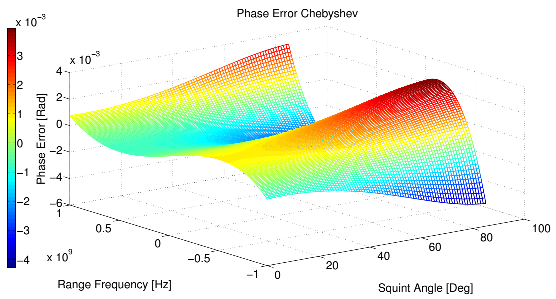
	Airborne		Spaceborne	
	Transmitter	Receiver	Transmitter	Receiver
Carrier Frequency	9.65 GHz		9.65 GHz	
Range Bandwidth	2 GHz		600 MHz	
Squint Angles	$0^\circ - 85^\circ$		$0^\circ - 45^\circ$	
Velocity	110 m/s	100 m/s	7600 m/s	7630 m/s
R_0	13.1 km	23.1 km	754 km	666 km

simulated configurations are reported, these configurations are similar to those used in [6].

The phase error for the airborne configuration is shown in Figure 1. Figure 1-a shows the phase error using the Taylor approximation while Figure 1-b shows the phase error using the Chebyshev approximation. The phase error obtained with the Chebyshev polynomial approximation resulted to be smaller than that obtained using the Taylor polynomial. The maximum error using Chebyshev resulted to be 74.6% less than that using the Taylor polynomial. The average absolute phase error resulted to be of 0.002 rad for the Taylor case while of 0.0012 rad for the Chebyshev approximation.



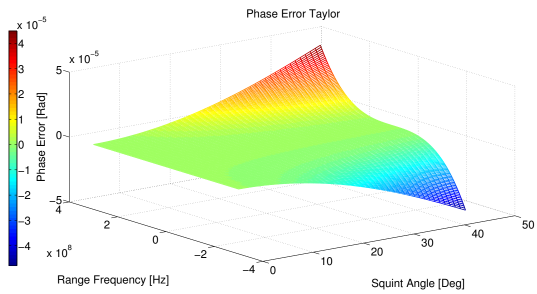
(a)



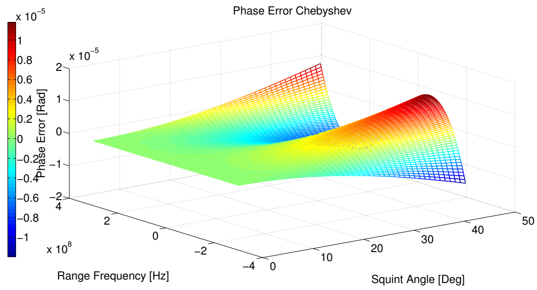
(b)

Figure 1: Phase Error for the airborne configuration for the a) Taylor and b) Chebyshev case.

The phase error for the spaceborne configuration is shown in Figure 2 Figure 2-a shows the phase error using the Taylor approximation while Figure 2-b shows the phase error using the Chebyshev approximation. The phase error obtained with the Chebyshev polynomial approximation resulted to be smaller than that obtained using the Taylor polynomial. The maximum error using Chebyshev resulted to be 75% less than that using the Taylor polynomial. The average absolute phase error resulted to be of 4.16×10^{-6} rad for the Taylor case while of 2.57×10^{-6} rad for the Chebyshev approximation.



(a)



(b)

Figure 2: Phase Error for the spaceborne onfiguration for the a) Taylor and b) Chebyshev case.

With these results we can state that it will be possible to improve the overall accuracy of the bistatic focussing algorithms, in particular monostatic focussing algorithms can be generalized to the bistatic case as in [6]. The main issue coming from the phase error is the constrain on the amount of data to process contemporary, in order keep an acceptable accuracy in [6] the processing was constrained to be done in blocks. With our approach this constrain still exists but with the same limit in terms of error bigger blocks can be adopted, improving the overall efficiency of the algorithms.

V. Conclusion

In this paper an improvement of the bistatic point target spectrum based on the 2D principle of stationary phase has been proposed. In our approach the double squared root bistatic frequency coupling term is expanded as series of Chebyshev polynomial, replacing the Taylor one proposed in literature. The new approach is easy to implement in existing algorithms exploiting this kind of model for the bistatic PTS. In addition the computational overload is negligible considering the general amount of computation required in the focussing process [9].
The resulting phase error using the Chebyshev based approach is reduced of about the 75% for both airborne and space-borne configurations. The direct advantage of this is the possibility to obtain a more efficient processing increasing the size of the data blocks keeping an acceptable phase error. Further analysis will be carried on analysing the improvement in focussing and efficiency of the bistatic RDA and CSA proposed in

[6] and the analysis of the residual range-azimuth coupling present in $\Phi_{RC}(f_\tau, f)$ and that will result probably reduced in $\hat{\Phi}_{RC}(f_\tau, f)$

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